UDC 533.6.011

# VARIATIONAL PROBLEMS OF OPTIMIZATION FOR EQUATIONS OF THE HYPERBOLIC TYPE IN THE PRESENCE OF BOUNDARY CONTROLS

PMM Vol. 39, №2, 1975, pp. 260-270
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(Received February 8, 1974)

We consider the problem of optimal control of the processes described by second order partial differential equations of the hyperbolic type. The present case differs from the problems analyzed [1−3]; here the control parameters appear in the right-hand side of the equation as well as in the boundary conditions. We construct the necessary conditions for a minimum of the functional, writing them in the expanded form. We illustrate the method by solving the problems of optimal loading of rods, and the problems include those with continuous and discontinuous Lagrange multipliers. We also give an example of a specific optimal control in the problem concerning the minimum of the total energy of the rod.

The most important part of the present paper which sets it appart from the work of the other authors who studied the analogous as well as the more general problems, concerns the investigation of discontinuities which the Lagrange multipliers undergo on the characteristics of the equation, and of the resulting changes in the terminal and boundary conditions. The discontinuities in the Lagrange multipliers were first studied in [4] in the course of solving variational problems of the gas dynamics. The above paper as well as a number of later papers, all made wide use of the discontinuities in the Lagrange multipliers on the characteristics, and of the discontinuities in the variables describing the state (shock waves and tangential discontinuities).

**1. Statement of the problem.** We consider the following partial differential equations and other relations defined in the two-dimensional space $\Omega$ $(a \leqslant x \leqslant b,$ $c \leqslant y \leqslant d)$:

$$L(z) = a_1 z_{xx} - a_2 z_{yy} + a_3 z_x + a_4 z_y = f(x, y, z, u) \qquad (1.1)$$

$$\psi_k(x, y, u) = 0, \quad k = 1, \ldots, r < m$$

Here $z_x$, $z_y$, $z_{xx}$ and $z_{yy}$ denote the first and second order partial derivatives of the function $z(x, y)$ which is to be determined, while $u = (u_1(x, y), \ldots, u_m(x, y))$ implies the $m$-dimensional vector of the piecewise continuous distributed controls.

The following initial and boundary conditions are assumed given:

$$z(a, y) = \varphi_1(y), \quad z_x(a, y) = \varphi_2(y) \qquad (1.2)$$

$$\varphi_c = \varphi_c[x, z(x, c), z_y(x, c)] = 0$$

$$\varphi_d = z_y(x, d) - g[x, z(x, d), v(x)] = 0$$

$$\psi_{d\tau}(x, v) = 0, \quad \tau = 1, \ldots, r_1 < m_1$$

where $v(x) = (v_1(x), \ldots, v_{m_1}(x))$ is the vector of the piecewise-continuous boundary controls. At the boundary $x = b$ the conditions

$$\chi_j [z(b, y_1^\circ), \ldots, z(b, y_{m_2}^\circ)] = 0, \quad j = 1, \ldots, r_2 \leqslant m_2 \tag{1.3}$$

connecting the values of the function $z$ at the fixed points $(b, y_\gamma^\circ)$, $\gamma = 1, \ldots, m_2$, $y_1^\circ = c$ and $y_{m2}^\circ = d$, may hold.

Let us pose the following optimal problem: to find amongst the surfaces satisfying (1.1) within the region $\Omega$ and the conditions (1.2), (1.3) at the boundaries of $\Omega$, a surface which minimizes the functional

$$J = \iint_\Omega f_0(x, y, z, u)\, dx\, dy + \int_a^b g_0[x, z(x, d), v(x)]\, dx + \tag{1.4}$$

$$\int_c^d \varphi_0[y, z(b, y), z_y(b, y), z_x(b, y)]\, dy + \chi_0[z(b, y_1^\circ), \ldots, z(b, y_{m_2}^\circ)]$$

The coefficients $a_i(x, y)$ appearing in Eq. (1.1) and the functions $f$, $\psi_k$, $\varphi_1$, $\varphi_2$, $\varphi_c$, $\psi_{d\tau}$, $g$, $\chi_j$, $f_0$, $g_0$, $\varphi_0$ and $\chi_0$ are assumed continuous together with their derivatives up to the third order inclusive.

**2. Necessary condition of stationarity.** The problem formulated above represents the two-dimensional Bolza problem of the variational calculus. Lemmas can be proved for this problem on the imbedding of the surface $E$ minimizing the functional (1.4) into one-parameter or a multi-parameter family of comparison surfaces. We can use these lemmas to prove the following necessary condition of stationarity of the functional $J$: the necessary condition for the functional $J$ to assume a minimum value on the surface $E$ is, that the equation

$$\Delta I = 0$$

holds on this surface. Here

$$I = \iint_\Omega [\lambda L(z) + H]\, dx\, dy + \int_a^b [\nu_c \varphi_c + \nu_d z_y(x, d) + h]\, dx + \tag{2.1}$$

$$\int_c^d \{\nu_1 [z(a, y) - \varphi_1] + \nu_2 [z_x(a, y) - \varphi_2] + \varphi_0\}\, dy +$$

$$\chi[z(b, y_1^\circ), \ldots, z(b, y_{m_2}^\circ)], \qquad H = f_0 - \lambda f + \sum_{k=1}^{r} \mu_k \psi_k$$

$$h = g_0 - \nu_d g + \sum_{\tau=1}^{r_1} \mu_{d\tau} \psi_{d\tau}, \qquad \chi = \chi_0 + \sum_{j=1}^{r_2} \rho_j \chi_j$$

where $\lambda(x, y)$, $\mu_k(x, y)$, $\mu_{d\tau}(x)$, $\nu_1(y)$, $\nu_2(y)$, $\nu_c(x)$, $\nu_d(x)$ and $\rho_j$ are the undetermined Lagrange multipliers. The first variation of $I$ is denoted by $\Delta I$

Repeating the manipulations described in detail in [2], we obtain the variation $\Delta I$ in the form

$$\Delta I = \sum_{i=1}^{n} \iint_{\omega_i} \Big\{\Big[M(\lambda) + \frac{\partial H}{\partial z}\Big] \delta z_i + \tag{2.2}$$

$$\sum_{k=1}^{m} \frac{\partial H}{\partial u_k} \delta u_{ki}\Big\} dx\,dy + \sum_{i=1}^{n} \sum_{j=1}^{\tau_i} \int_{S_{ij}} \{A\lambda \Delta z_N +$$

$$(-A\lambda_N - 2B\lambda_s + G_2\lambda)\Delta z + [2B\lambda z_{sN} + A'\lambda z_{ss} + G_1\lambda z_N +$$

$$(-a_3 n_2 + a_4 n_1 - 2B\rho^{-1})\lambda z_s + (A\lambda_N + 2B\lambda_s -$$

$$G_2\lambda) z_N + H]\,\delta N\}\,ds + \sum_{i=1}^{n} \sum_{j=1}^{\tau_i} [B\lambda\,dz - B\lambda(z_x\,dx + z_y\,dy)]_{M_{ij}}^{M_{ij+1}} +$$

$$\sum_{j=k_a+1}^{k_a+m_a} \int_{y_{j-1}}^{y_j} [\nu_{1j}\Delta z_j(a, y) + \nu_{2j}\Delta z_{jx}(a, y)]\,dy +$$

$$\sum_{j=k_c+1}^{k_c+m_c} \int_{x_{j-1}}^{x_j} \nu_{cj}\left[\frac{\partial\varphi_c}{\partial z}\Delta z_j(x, c) + \frac{\partial\varphi_c}{\partial z_y}\Delta z_{jy}(x, c)\right] dx +$$

$$\sum_{j=k_d+1}^{k_d+m_d} \int_{x_{j-1}}^{x_j} \Big[\nu_{dj}\Delta z_{jy}(x, d) + \frac{\partial h}{\partial z}\Delta z_j(x, d) +$$

$$\sum_{t=1}^{m_1} \frac{\partial h}{\partial v_t}\Delta v_{tj}(x)\Big]\,dx + \sum_{j=k_b+1}^{k_b+m_b} \Big\{ \int_{y_{j-1}}^{y_j} \Big[\Big(\frac{\partial\varphi_0}{\partial z} - \frac{d}{dy}\frac{\partial\varphi_0}{\partial z_y}\Big)\Delta z_j(b, y) +$$

$$\frac{\partial\varphi_0}{\partial z_x}\Delta z_{jx}(b, y)\Big]\,dy + \Big[\frac{\partial\varphi_0}{\partial z_y}dz(b, y) + \Big(\varphi_0 - \frac{\partial\varphi_0}{\partial z_y} z_y\Big)dy\Big]_{y_{j-1}}^{y_j}\Big\} +$$

$$\sum_{\gamma=1}^{m_2} \frac{\partial\chi}{\partial z(b, y_\gamma^\circ)}\,dz(b, y_\gamma^\circ) = 0$$

$$M(\lambda) = (a_1\lambda)_{xx} - (a_2\lambda)_{yy} - (a_3\lambda)_x - (a_4\lambda)_y$$

$$A = a_1 n_1^2 - a_2 n_2^2, \qquad B = -(a_1 + a_2)n_1 n_2, \qquad A' = a_1 n_2^2 - a_2 n_1^2$$

$$G_1 = a_3 n_1 + a_4 n_2 + A'\rho^{-1}, \qquad G_2 = G_1 - A_N - 2B_s - A\rho^{-1}$$

Here $n_1$ and $n_2$ are the direction cosines of the normal to the curve in question, $\rho$ is its radius of curvature, $n$ is the number of the elementary regions $w_i$, $\tau_i$ is the number of corner points on the boundary $S_i$ of each elementary region; $m_a$, $m_c$, $m_d$ and $m_b$ denote the numbers of the elementary regions with the lines $x = a$, $y = c$, $y = d$ and $x = b$ as the corresponding boundaries $k_a$, $k_c$, $k_d$ and $k_b$ are the numbers from which the sequential numbering of each of the above regions begins, $A_N$ and $B_s$ are the derivatives normal and tangential with respect to the arc of the curve under consideration.

Using (2.2) and applying the usual arguments, we can conclude that the coeffcient accompanying each variation must be equal to zero. Consequently, we have the following equations

$$M(\lambda) = -\partial H / \partial z, \qquad \partial H / \partial u_k = 0, \qquad k = 1, \ldots, m \tag{2.3}$$

which hold in each of the elementary regions $\omega_i$.

The conditions

$$\lambda_N^- = \lambda_N^+, \quad \lambda^- = \lambda^+, \quad H^- = H^+$$

must hold along the noncharacteristic boundary lines of $\omega_i$ lying inside the region $\Omega$, while along the characteristic lines we have

$$-2B\,(\lambda^- - \lambda^+)_s + G_2\,(\lambda^- - \lambda^+) = 0$$

The following condition must hold at each corner point lying inside the region $\Omega$ [2]:

$$\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0$$

Analyzing the terms containing the variations $\Delta z_y\,(x,\ c)$ and $\Delta z\,(x,\ c)$ for the segment $y = c,\ a \leqslant x \leqslant b$ of the boundary, we arrive at the following relations:

$$a_2\lambda = \nu_c \partial\varphi_c \,/\, \partial z_y \qquad (y = c) \qquad (2.4)$$
$$a_2\lambda_y + (a_4 + a_{2y})\lambda = -\nu_c \partial\varphi_c \,/\, \partial z$$

for each boundary region with $y = c$ as the boundary. When $\partial\varphi_c \,/\, \partial z_y \neq 0$, the following relation must hold at each corner point lying on the boundary $y = c$ and not coincident with $(a,\ c)$ and $(b,\ c)$ :

$$\lambda\,(x - 0,\ c) - 2\lambda\,(x,\ c) + \lambda\,(x + 0,\ c) = 0$$

At the boundary $y = d$ we have

$$a_2\lambda = \nu_d, \quad a_2\lambda_y + (a_4 + a_{2y})\,\lambda = \partial h \,/\, \partial z \qquad (y = d) \qquad (2.5)$$
$$\partial h \,/\, \partial v_t = 0, \quad t = 1,\ \ldots,\ m_1$$

for each elementary region $\omega_i$ with $y = d$ as the boundary. The following relation must hold at each corner point lying on the boundary $y = d$ and not coincident with $(a,\ d)$ and $(b,\ d)$ :

$$\lambda\,(x - 0,\ d) - 2\lambda\,(x,\ d) + \lambda\,(x + 0,\ d) = 0$$

At the boundary $x = a$ the relation

$$a_1\lambda = -\nu_1, \quad a_1\lambda_x - (a_3 - a_{1x})\lambda = \nu_2, \quad x = a$$

holds along those boundaries of the elementary regions which coincide with $x = a$. It is possible that at the corner points lying on the boundary $x = a$

$$\lambda\,(a,\ y - 0) - 2\lambda\,(a,\ y) + \lambda\,(a,\ y + 0) \neq 0$$

i.e. the Lagrange multiplier $\lambda$ may undergo a discontinuity at the boundary $x = a$ on one or two characterisitcs of Eq. (2.3).

The following conditions obtain for the segment $x = b,\ c < y < d$ of the boundary, in each elementary region with $x = b$ as the boundary

$$a_1\lambda = -\partial\varphi_0 \,/\, \partial z_x, \quad x = b \qquad (2.6)$$
$$a_1\lambda_x - (a_3 - a_{1x})\,\lambda = \partial\varphi_0 \,/\, \partial z - (d \,/\, dy)\,(\partial\varphi_0 \,/\, \partial_{zy})$$

At each corner point on the boundary $x = b$ and not coincident with $(b,\ c)$ or $(b,\ d)$, we have

$$\lambda\,(b,\ y) = {}^1\!/_2\,\lambda\,(b,\ y - 0) - {}^1\!/_2\,\lambda\,(b,\ y + 0) + (a_1 a_2)^{-1/2}\,[\partial\chi \,/ \qquad (2.7)$$
$$/\, \partial z\,(b,\ y) - \partial\varphi_0^{\,+} \,/\, \partial z_y + \partial\varphi_0^{\,-} \,/\, \partial z_y] \qquad (x = b)$$

$$\partial\chi / \partial z(b, y) = \begin{cases} \partial\chi / \partial z(b, y_\gamma^\circ), & y = y_\gamma^\circ \\ 0, & y \neq y_\gamma^\circ \end{cases}$$

The following relation must hold at the point $(b, d)$ :

$$\lambda(b-0, d) = \lambda(b, d-0) - (a_1a_2)^{-1/2}[\partial\chi / \partial z(b, d) + \partial\varphi_0 / \partial z_y \mid_{y=d}] \tag{2.8}$$

while at the point $(b, c)$ with $\partial\varphi_c / \partial z_y \neq 0$ we have

$$\lambda(b-0, c) = \lambda(b, c+0) - (a_1a_2)^{-1/2}[\partial\chi/\partial z(b, c) - \partial\varphi_0 / \partial z_y|_{y=c}] \tag{2.9}$$

In addition to the conditions obtained above, another condition analogous to (4.14) of [2] must hold at each corner point $(b, y_j)$. For the present case this condition has the form

$$\sum_{\alpha=1}^{2}\sum_{i=1}^{k_\alpha}\Theta_{\alpha i}\left(\vartheta_{\alpha i} + \int_{x_{\alpha i}}^{x_{\alpha i+1}}\theta_{\alpha i}\,dx\right) = \left[a_1\lambda(z_x^+ - z_k^-) + \varphi_0^+ - \varphi_0^- - \frac{\partial\varphi_0^+}{\partial z_y}z_y^+ + \frac{\partial\varphi_0^-}{\partial z_y}z_y^-\right] = 0, \quad x = b,\ y = y_j$$

$$\vartheta_{\alpha i} = \begin{cases} 0, & i = 1, \alpha = 1, 2 \quad \text{or} \quad i = 2, \ldots, k_\alpha,\ k_\alpha - i + \alpha\text{- odd} \\ \dfrac{[g_0^- - g_0^+ - \frac{1}{2}(\nu_d^- + \nu_d^+)(g^- - g^+)]\,|_{x_{\alpha i}}}{F_{2x}(x_{\alpha i}, d)}, & i = 2, \ldots, k_\alpha,\ k_\alpha - i + \alpha\text{- even} \end{cases}$$

All the remaining quantities have been determined in [2].

**3. The necessary Weierstrass conditions for a strong minimum.** Let us now establish the necessary Weierstrass conditions for the strong minimum. These conditions will include a condition at the internal points of the region $\Omega$, and a condition at the points of the boundary $y = d$. The first of these conditions was obtained in [3] and it has the following form:

$$H(x, y, z, U, \mu, \lambda) \geqslant H(x, y, z, u, \mu, \lambda) \tag{3.1}$$

where $U$ denotes any admissible control and $u$ denotes the control which produces the required surface.

The second necessary Weierstrass condition of the strong minimum can be represented by the inequality

$$h(x, z, V, \nu_d, \mu_d) \geqslant h(x, z, v, \nu_d, \mu_d) \tag{3.2}$$

$$V = (V_1, \ldots, V_{m_1}), \quad \mu_d = (\mu_{d1}, \ldots, \mu_{dr_1})$$

where $V$ denotes the admissible control parameters, $\mu_d$, are the Lagrange multipliers and $v$ is the control yielding the required surface. The condition (3.2) is proved below in Sect. 6.

Thus we see that for the surface to minimize the functional $J$ it is necessary that the condition of stationarity (2.2) holds, that at each internal point of the elementary regions the Weierstrass condition (3.1) is valid, and that the Weierstrass condition (3.2) of strong minimum holds at each point of the boundary $y = d$ which does not coincide with the corner point.

**4. Construction of optimal loading of a rod by a concentrated force.** A rod of constant cross section is given. It is clamped at one end, and a concentrated force is applied at its other end. The equation of motion of the rod in the region $\Omega$ $(0 \leqslant x \leqslant T,\ 0 \leqslant y \leqslant l)$ has the form

$$z_{xx} - a^2 z_{yy} = 0 \tag{4.1}$$

Let the initial and boundary conditions be given by

$$z(0,\ y) = z_x(0,\ y) = 0 \tag{4.2}$$

$$z(x,\ 0) = 0,\ z_y(x,\ l) = v_1(x), \quad |v_1(x)| \leqslant F \tag{4.3}$$

To solve the optimal problem, we must pass to the open domain of definition of $v_1(x)$. This can be done by introducing an additional control $v_2(x)$ [3] and constructing the relation

$$\psi = F^2 - v_1^2 - v_2^2 = 0 \tag{4.4}$$

We formulate the optimization problem as follows: to find the control $v_1(x)$ which imparts to the functional (1.4) at $f_0 \equiv 0$ a minimum value. In accordance with our assertions we introduce the undetermined Lagrange coefficients $\lambda(x,\ y)$, $\nu_1(y)$, $\nu_2(y)$, $\nu_c(x)$, $\nu_d(x)$ and $\mu_d(x)$. For $\lambda$ the Euler equation has the form

$$\lambda_{xx} - a^2 \lambda_{yy} = 0 \tag{4.5}$$

The formulas (2.4) and (2.5) yield the boundary conditions

$$\begin{gathered} \lambda(x,\ 0) = 0, \quad a\lambda_y(x,\ l) = -\partial g_0 / \partial z \\ \nu_c(x) = a\lambda_y(x,\ 0), \quad \nu_d(x) = a\lambda(x,\ l) \end{gathered} \tag{4.6}$$

and the equations

$$-\nu_d - 2\mu_d \nu_1 = 0, \quad -2\mu_d v_2 = 0 \tag{4.7}$$

connecting the controls $v_1$ and $v_2$. From the necessary condition of Weierstrass (3.2) we can obtain the following control in the form:

$$v_1(x) = F \operatorname{sign} \nu_d(x) \tag{4.8}$$

Let us consider the functional

$$J = \int_0^T z(x,\ l)\, dx \tag{4.9}$$

Formulas (2.6) yield the terminal conditions

$$\lambda(T,\ y) = \lambda_x(T,\ y) = 0, \quad 0 \leqslant y < l \tag{4.10}$$

Solving Eq. (4.5) with the terminal conditions (4.10) and the boundary conditions (4.6), we can find $\lambda(x,\ y)$ and $\nu_d(x)$

$$\nu_d(x) = \frac{4l}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left[\cos \frac{(2k+1)\pi a(x-T)}{2l} - 1\right] \tag{4.11}$$

From (4.11) it follows that $\nu_d(x) \leqslant 0$, therefore $v_1(x) = -F$. Substituting the control obtained into the boundary conditions (4.3) and solving Eq. (4.1), we find $z(x,\ y)$ and hence the functional

$$J = -\frac{Fl^3}{a}\left[n + \frac{ax_1}{2l} - (-1)^n \left(1 - \frac{ax_1}{2l}\right)\right]$$

$$n = \operatorname{entier}[aT / (2l)], \quad x_1 = T - 2nl / a$$

Let us now consider the functional

$$J = \int_0^l z_x(T, y)\, dy \tag{4.12}$$

In this case the terminal and boundary conditions for the multiplier $\lambda$ assume the form

$$\lambda(T, y) = -1, \quad \lambda_x(T, y) = \lambda(x, 0) = \lambda_y(x, l) = 0 \tag{4.13}$$

from which it is evident that at the point $x = T, y = 0$, the multiplier $\lambda$ has a discontinuity. We find the solution of (4.5) with the discontinuous initial conditions by employing the procedure described in detail in [3]. Analyzing the sign of the Lagrange multiplier $\nu_d(x)$ and the Weierstrass condition (4.8) we obtain the following formulas:

$$v_1(x) = (-1)^{n-1} F, \quad x_{i-1} < x < x_i, \quad i = 1, \ldots, n+1$$
$$x_0 = 0, \quad x_i = T - l / a - 2(n-i)\, l / a, \quad x_{n+1} = T$$
$$n = \operatorname{entier}[1 + aT / (2l)]$$

$$J = \begin{cases} -FTa^2, & 0 \leqslant T < l / a \\ Fla(-2n + 1 - ax_1 / l), & T \geqslant l / a \end{cases}$$

Let the functional $J$ be written in the form

$$J = -\alpha l z(T, l) + \int_0^l z(T, y)\, dy \tag{4.14}$$

In this case the terminal conditions for $\lambda(x, y)$ are obtained from the formulas (2.6) and (2.8)

$$\lambda(T, y) = 0, \quad \lambda_x(T, y) = 1, \quad 0 \leqslant y < l, \quad \lambda(T, l) = \alpha l / a \tag{4.15}$$

The boundary conditions remain unchanged. From the conditions (4.15) and the boundary conditions (4.14) it is clear that $\lambda$ has a discontinuity at the point $(x = T, y = l)$. Using the procedure for obtaining a discontinuous solution given in [3], we find $\lambda(x, y)$ and the control $v_1(x)$

$$v_1(x) = (-1)^{n-i} F, \quad x_{i-1} < x < x_i \tag{4.16}$$

for $\alpha < 0$ and

$$v_1(x) = \begin{cases} -(-1)^{n-i} F, & x_i - \alpha l / a < x < x_i \\ (-1)^{n-i} F, & x_{i-1} + \alpha l / a < x < x_i - \alpha l / a \\ -(-1)^{n-i} F, & x_{i-1} < x < x_{i-1} + \alpha l / a \end{cases} \tag{4.17}$$

$$i = 1, \ldots, n+1, \quad n = \operatorname{entier}[aT / (2l)], \quad x_0 = 0,$$
$$x_i = T - 2l(n + 1 - i) / a$$

for $0 < \alpha < 1$.

With the control (4.16) the value of the functional $J$ is $J = J_0$, while with (4.17) we have $J = J_0 + J_\alpha$, where

$$J_0 = \begin{cases} Fl^2[-n + 2\alpha n + \alpha a x_1 / l - a^2 x_1^2 / (2l)^2], & 0 \leqslant x_1 \leqslant l / a \\ Fl^2[-n - 1 + 2\alpha n + \alpha a x_1 / l + {}^1/_2 (2 - a x_1 / l)^2], & l / a \leqslant x_1 < 2l / a \end{cases}$$

$$J_\alpha = \begin{cases} Fl^2[-2n\alpha^2 - 2\alpha a x_1 / l + a^2 x_1^2 / l^2], & 0 \leqslant x_1 \leqslant \alpha l / a \\ Fl^2(-2n\alpha^2 - \alpha^2), & \alpha l / a \leqslant x_1 \leqslant 2l / a - \alpha l / a \\ Fl^2[-2n\alpha^2 - 2\alpha^2 + 4\alpha - 2\alpha a x_1 / l - (2 - a x_1 / l)^2], & 2l / a - \alpha l / a \leqslant x < 2l / a \end{cases}$$

**5. Example of a particular control.** A problem on the minimum of the dynamic coefficient was solved in [5] and a load under which this coefficient is equal to unity, was found. Evidently, the problem on the minimum of the dynamic coefficient for a one-mass system is equivalent to the problem of the loading under which the total energy of the system becomes minimum at the specified time. For the system with distributed parameters, we can also pose a problem in which the loading is such that the oscillations about the statistical equilibrium are as small as possible. In analogy with the one-mass system we shall use the total energy of the system as the criterion of optimality.

Let us formulate the following problem: to load a rod of constant cross section clamped at one end, with a force $P$ applied at the other end for a time $0 \leqslant x \leqslant T$ in such a manner, that the total energy of the rod is minimum at the instant $T$. The problem is described by Eqs. (4.1)–(4.3) in which the last inequality is replaced by

$$0 \leqslant v_1(x) \leqslant F = P / (\sigma E)$$

where $\sigma$ is the area of transverse cross section and $E$ is the Young's modulus. The total energy of the rod is proportional to the functional

$$J = \int_0^l \{z_x^2(T, y) + a^2 [z_y(T, y) - z_{cm\,y}(y)]^2\}\, dy$$

where $z_{cm}(y)$ denotes the statistical displacement of the rod under the action of the force $P$.

The Lagrange multiplier $\lambda(x, y)$ satisfies Eq. (4.5) together with the boundary and terminal conditions

$$\lambda(x, 0) = \lambda_y(x, l) = 0 \qquad (5.1)$$

$$\lambda(T, y) = -2z_x(T, y), \quad 0 \leqslant y < l \qquad (5.2)$$

$$\lambda(T, l) = \lambda(T, l-0) - 2a[z_y(T, l) - z_{cm\,y}(l)]$$

$$\lambda_x(T, y) = -2a^2[z_{yy}(T, y) - z_{cm\,yy}(y)], \quad 0 < y < l$$

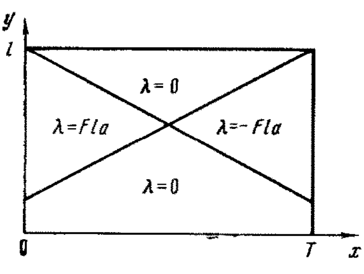

Fig. 1

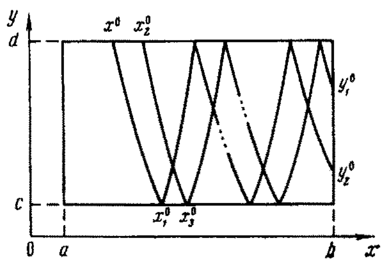

Fig. 2

The above problem was solved using the gradient method with $F = a = l = 1$, and the discontinuous part of the multiplier $\lambda(x, y)$ governed by the second relation of (5.2) was computed separately. This gave the solution $v_1(x) = {}^1\!/_2 F$. It was found that such a control gives an exact solution of the problem posed above. We shall show that the control satisfies all necessary conditions obtained previously. Substituting $v_1(x) = {}^1\!/_2 F$ into the boundary conditions (4.3) of Eq. (4.1) and solving, we obtain

$$z(x, y) = \frac{4Fl^2}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left[1 - \cos\frac{(2k+1)\pi a x}{2l}\right] \sin\frac{(2k+1)\pi y}{2l}$$

Solution of (4.5) gives

$$\lambda(x, y) = -\frac{4Fla}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left[\sin\frac{(2k+1)\pi a x}{2l} - \right.$$

$$\sin\frac{(2k+1)\pi a(T-x)}{2l}\Big|\sin\frac{(2k+1)\pi y}{2l}+\begin{cases}0, & x,y\in\Omega\setminus\omega\\ 2Fla, & x,y\in\omega\end{cases}$$

Figure 1 shows the multiplier $\lambda(x, y)$ in the region $\Omega$. We see that $v_d(x) = a\lambda(x, l) \equiv 0$. hence the control $v_1(x)$ is a particular one. It can assume any admissible values. The functional is given by the formula

$$J(T)=\begin{cases}Fl^2a^2[1-aT/(2l)], & T\leqslant 2l/a\\ 0, & T>2l/a\end{cases}$$

i.e. in the period $T = T^* = 2l/a$ the rod can be loaded in such a manner, that beginning from the instant $T^*$, it will be in the state of static equilibrium.

6. **Proof of the condition** (3.2). The necessary Weierstrass condition (3.2) of strong minimum can be obtained as follows. We take the normal surface $E$ which minimizes the functional $J$. We choose at the boundary $y = d$ the points $x^\circ$ and $x_2^\circ = x^\circ + e$ not coinciding with the corner points in such a manner that the segment $[x^\circ, x_2^\circ]$ belongs to any one of the elementary regions $\omega_i$, and draw from the points $(x^\circ, d)$ and $(x_2^\circ, d)$ the saw-tooth curves $Q_1$ and $Q_2$ consisting of the segments of the characteristics of the families $C$ and $D$ (see Fig. 2). We construct the following admissible family of functions:

$$\begin{aligned}&z(x, y),\quad u_k(x, y),\quad v_t(x),\quad x, y\in\Omega_1,\quad k=1,\ldots,m \qquad (6.1)\\ &Z(x, y),\quad u_k(x, y),\quad V_t(x),\quad x, y\in\Omega_2\\ &z(e, x, y),\quad u_k(x, y),\quad v_t(x),\quad x, y\in\Omega_3,\quad t=1,\ldots,m_1\end{aligned}$$

which includes the surface $E$ at $e = 0$. Here $V_t(x)$ denote any controls satisfying the last equations of (1.2). The following conditions hold at the boundaries of the regions $\Omega_1$, $\Omega_2$ and $\Omega_3$:

$$z(x, y)|_D = Z(x, y)|_D,\qquad Z(x, y)|_{D_e} = z(e, x, y)|_{D_e} \qquad (6.2)$$

while on the remaining segments of the lines $Q_1$ and $Q_2$ the conditions

$$z(e, x, y+0) = z(e, x, y-0) \qquad (6.3)$$

hold. The variations of the family (6.1) over the parameter $e$ on the surface $E$ ($e = 0$) are

$$\begin{aligned}&\delta z(x, y) = (dz/\partial e)_{e=0}\,de = 0,\quad x, y\in\Omega_1\cup\Omega_2\\ &\delta u_k(x, y) = (\partial u_k/\partial e)_{e=0}\,de = 0,\quad k=1,\ldots,m\\ &\delta v_t(x) = (\partial v_t/de)_{e=0}\,de = 0,\quad t=1,\ldots,m_1\end{aligned}$$

because altering the boundary controls at the point $x_2^\circ$ changes $z(x, y)$ in the region $\Omega_3$, and the functions $u_k(x, y)$ and $v_t(x)$ are independent of the parameter $e$. On the characteristic $D_e$ the variation $\delta z$ satisfies the condition

$$[(Z_N - z_N)\delta N]_{D_e} = \delta z|_{D_e},\qquad e = 0$$

which is obtained by differentiating the second condition of (6.2) with respect to $e$. The same differentiation of (6.3) yields

$$[(z_N^- - z_N^+)\delta N]_{Q_2} = (\delta z^+ - \delta z^-)|_{Q_2},\qquad \delta z^+|_{Q_1} = \delta z^-|_{Q_1}$$

since the coordinates $x$ and $y$ along the line $Q_1$ are independent of $e$. Substituting the functions (6.1) into the functional (2.2), differentiating in $e$ for $e = 0$, we obtain

$$\Delta I = \left(\frac{\partial I}{\partial e}\right)_{e=0}de = \pm\int_{Q_2}[L^+(z) - L^-(z) + H^+ - H^-]\,\delta N\,ds + \sum_{i=1}^{n'}\left\{\iint_{\omega_i}\Big[M(\lambda) +\right.$$

$$\left.\frac{\partial H}{\partial z}\right]\delta z_i\,dx\,dy + \oint [A\lambda\delta z_N + (-A\lambda_N - 2B\lambda_s + G_2\lambda)\,\delta z]\,ds + \sum_{j=1}^{\tau_i'} [B\lambda\delta z]_{M_{ij}}^{M_{ij+1}} +$$

$$\sum_{j=k_c'+1}^{k_c'+m_c'} \int_{x_{j-1}}^{x_j} \nu_{cj}\left[\frac{\partial\varphi_c}{\partial z}\,\delta z_j(x, c) + \frac{\partial\varphi_c}{\partial z_y}\,\delta z_{jy}(x, c)\right] dx + \sum_{j=k_d'+1}^{k_d'+m_d'} \int_{x_{j-1}}^{x_j} \left[\nu_{dj}\delta z_{jy}(x, d) + \right.$$

$$\left.\left(\frac{\partial h}{\partial e}\right)_{e=0} de\right] dx + [\nu_d(x)\,z_y(x, d) + h]\,\delta x\,\Big|_{x_{j-1}}^{x_j}\Big\} + \sum_{j=k_b'+1}^{k_b'+m_b'} \int_{y_{j-1}}^{y_j} \left(\frac{\partial\varphi_0}{\partial e}\right)_{e=0} de\,dy +$$

$$[(\varphi_0^- - \varphi_0^+)\,\delta y_2^0] + \left(\frac{\partial\chi}{\partial e}\right)_{e=0} de$$

$$x_{k_c'} = x_1^c, \quad x_{k_c'+m_c'} = b, \quad x_{k_d'} = x^\circ, \quad x_{k_d'+m_d'} = b, \quad x_{k_b'} = c, \quad x_{k_b'+m_b'} = d$$

Here the primed quantities denote integers determined in the same manner as the unprimed quantites, but referring to the region $\Omega_3$, the plus (minus) sign is taken for the characteristics with the derivative $dy / dx$ greater (smaller) than zero.

First, we note that the Lagrange multiplier $\lambda$ and the function $L(z) + H$ are both continuous along the line $Q_2$. This is because $Q_2$ is not the line of the surface $E$ minimizing the functional $J$. When $e = 0$ we have $\delta z\,|_{Q_2} = (z_N^+ - z_N^-)\delta N\,|_{Q_2} = 0$. Therefore the integrals on the line $Q_2$ as well as the terms at the points lying on these lines all vanish.

Using the condition (2.2) of stationarity of the functional $J$, we obtain

$$\Delta I = \sum_{j=k_d'}^{k_d'+m_d'} [h(x_j, z, v^-) - h(x_j, z, v^+)]\,\delta x_j \qquad (6.4)$$

We have $v^- = v^+$ at all points $x_j$ except $x_{k_d'} = x^\circ$, therefore Eq. (6.4) becomes

$$\Delta I = [h(x^\circ, z, V) - h(x^\circ, z, v)]de$$

The surface $E$ minimizes the functional $J$, therefore

$$\Delta I \geqslant 0 \qquad (6.5)$$

Since the point $x^\circ$ is chosen arbitrarily and $de > 0$ by definition, we can write (6.5) in the form

$$h(x, z, V, \nu_d, \mu_d) - h(x, z, v, \nu_d, \mu_d) \geqslant 0 \qquad (6.6)$$

where $V$ are the admissible control parameters connected by the last set of relations in (1.2) and $\mu_d$ are the Lagrange multipliers. As the controls $V_t(x)$ have no additional conditions imposed on them, we can speak of the necessary Weierstrass condition of the strong minimum (6.6).

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Translated by L. K.